Tuesday, FEBRUARY 12, 2008

9th Annual American Mathematics Contest 10

AMC 10
CONTEST A

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 10 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, and erasers. No calculators are allowed. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have 75 **MINUTES** to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

Students who score 120 or above or finish in the top 1% on this AMC 10 will be invited to take the 26th annual American Invitational Mathematics Examination (AIME) on Tuesday, March 18, 2008 or Wednesday, April 2, 2008. More details about the AIME and other information are on the back page of this test booklet.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

1. A bakery owner turns on his doughnut machine at 8:30 AM. At 11:10 AM the machine has completed one third of the day's job. At what time will the doughnut machine complete the job?

   (A) 1:50 PM **(B)** 3:00 PM **(C)** 3:30 PM **(D)** 4:30 PM **(E)** 5:50 PM

2. A square is drawn inside a rectangle. The ratio of the width of the rectangle to a side of the square is 2:1. The ratio of the rectangle's length to its width is 2:1. What percent of the rectangle's area is inside the square?

   (A) 12.5 **(B)** 25 **(C)** 50 **(D)** 75 **(E)** 87.5

3. For the positive integer n, let $<n>$ denote the sum of all the positive divisors of n with the exception of n itself. For example, $<4> = 1+2 = 3$ and $<12> = 1+2+3+4+6 = 16$. What is $<<<6>>>$?

   (A) 6 **(B)** 12 **(C)** 24 **(D)** 32 **(E)** 36

4. Suppose that $\frac{2}{3}$ of 10 bananas are worth as much as 8 oranges. How many oranges are worth as much as $\frac{1}{2}$ of 5 bananas?

   (A) 2 **(B)** $\frac{5}{2}$ **(C)** 3 **(D)** $\frac{7}{2}$ **(E)** 4

5. Which of the following is equal to the product

$$\frac{8}{4} \cdot \frac{12}{8} \cdot \frac{16}{12} \cdots \cdots \frac{4n+4}{4n} \cdots \cdots \frac{2008}{2004}?$$

   (A) 251 **(B)** 502 **(C)** 1004 **(D)** 2008 **(E)** 4016

6. A triathlete competes in a triathlon in which the swimming, biking, and running segments are all of the same length. The triathlete swims at a rate of 3 kilometers per hour, bikes at a rate of 20 kilometers per hour, and runs at a rate of 10 kilometers per hour. Which of the following is closest to the triathlete's average speed, in kilometers per hour, for the entire race?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

7. The fraction

$$\frac{\left(3^{2008}\right)^2 - \left(3^{2006}\right)^2}{\left(3^{2007}\right)^2 - \left(3^{2005}\right)^2}$$

   simplifies to which of the following?

   (A) 1 **(B)** $\frac{9}{4}$ **(C)** 3 **(D)** $\frac{9}{2}$ **(E)** 9

8. Heather compares the price of a new computer at two different stores. Store A offers 15% off the sticker price followed by a \$90 rebate, and store B offers 25% off the same sticker price with no rebate. Heather saves \$15 by buying

the computer at store A instead of store B. What is the sticker price of the computer, in dollars?

(A) 750 **(B)** 900 **(C)** 1000 **(D)** 1050 **(E)** 1500

9. Suppose that

$$\frac{2x}{3} - \frac{x}{6}$$

is an integer. Which of the following statements must be true about x?

(A) It is negative. **(B)** It is even, but not necessarily a multiple of 3.

(C) It is a multiple of 3, but not necessarily even.

(D) It is a multiple of 6, but not necessarily a multiple of 12.

(E) It is a multiple of 12.

10. Each of the sides of a square S_1 with area 16 is bisected, and a smaller square S_2 is constructed using the bisection points as vertices. The same process is carried out on S_2 to construct an even smaller square S_3. What is the area of S_3?

(A) $\frac{1}{2}$ **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

11. While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing toward the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 10

12. In a collection of red, blue, and green marbles, there are 25% more red marbles than blue marbles, and there are 60% more green marbles than red marbles. Suppose that there are r red marbles. What is the total number of marbles in the collection?

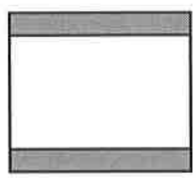
(A) $2.85r$ **(B)** $3r$ **(C)** $3.4r$ **(D)** $3.85r$ **(E)** $4.25r$

13. Doug can paint a room in 5 hours. Dave can paint the same room in 7 hours. Doug and Dave paint the room together and take a one-hour break for lunch. Let t be the total time, in hours, required for them to complete the job working together, including lunch. Which of the following equations is satisfied by t?

(A) $\left(\frac{1}{5}+\frac{1}{7}\right)(t+1)=1$ **(B)** $\left(\frac{1}{5}+\frac{1}{7}\right)t+1=1$ **(C)** $\left(\frac{1}{5}+\frac{1}{7}\right)t=1$

(D) $\left(\frac{1}{5}+\frac{1}{7}\right)(t-1)=1$ **(E)** $(5+7)t=1$

14. Older television screens have an aspect ratio of 4:3. That is, the ratio of the width to the height is 4:3. The aspect ratio of many movies is not 4:3, so they are sometimes shown on a television screen by "letterboxing" — darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of 2:1 and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?



(A) 2 **(B)** 2.25 **(C)** 2.5 **(D)** 2.7 **(E)** 3

15. Yesterday Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?

(A) 120 **(B)** 130 **(C)** 140 **(D)** 150 **(E)** 160

16. Points A and B lie on a circle centered at O, and $\angle AOB = 60°$. A second circle is internally tangent to the first and tangent to both $\overline{OA}$ and $\overline{OB}$. What is the ratio of the area of the smaller circle to that of the larger circle?

(A) $\frac{1}{16}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{4}$

17. An equilateral triangle has side length 6. What is the area of the region containing all points that are outside the triangle and not more than 3 units from a point of the triangle?

(A) $36+24\sqrt{3}$ **(B)** $54+9\pi$ **(C)** $54+18\sqrt{3}+6\pi$ **(D)** $\left(2\sqrt{3}+3\right)^2\pi$

(E) $9\left(\sqrt{3}+1\right)^2\pi$

18. A right triangle has perimeter 32 and area 20. What is the length of its hypotenuse?

(A) $\frac{57}{4}$ **(B)** $\frac{59}{4}$ **(C)** $\frac{61}{4}$ **(D)** $\frac{63}{4}$ **(E)** $\frac{65}{4}$

19. Rectangle $PQRS$ lies in a plane with $PQ = RS = 2$ and $QR = SP = 6$. The rectangle is rotated $90°$ clockwise about R, then rotated $90°$ clockwise about the point that S moved to after the first rotation. What is the length of the path traveled by point P?

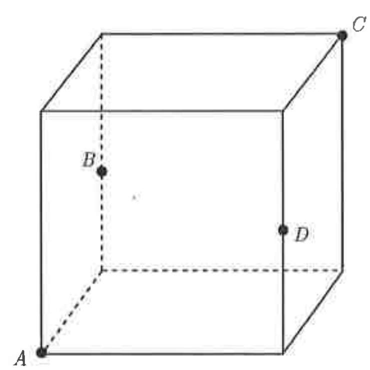
(A) $\left(2\sqrt{3}+\sqrt{5}\right)\pi$ **(B)** 6π **(C)** $\left(3+\sqrt{10}\right)\pi$ **(D)** $\left(\sqrt{3}+2\sqrt{5}\right)\pi$

(E) $2\sqrt{10}\pi$

20. Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$ and diagonals intersecting at K. Suppose that $AB = 9$, $DC = 12$, and the area of $\triangle AKD$ is 24. What is the area of trapezoid $ABCD$?

(A) 92 **(B)** 94 **(C)** 96 **(D)** 98 **(E)** 100

21. A cube with side length 1 is sliced by a plane that passes through two diagonally opposite vertices A and C and the midpoints B and D of two opposite edges not containing A or C, as shown. What is the area of quadrilateral $ABCD$?



(A) $\frac{\sqrt{6}}{2}$ **(B)** $\frac{5}{4}$ **(C)** $\sqrt{2}$ **(D)** $\frac{3}{2}$

(E) $\sqrt{3}$

22. Jacob uses the following procedure to write down a sequence of numbers. First he chooses the first term to be 6. To generate each succeeding term, he flips a fair coin. If it comes up heads, he doubles the previous term and subtracts 1. If it comes up tails, he takes half of the previous term and subtracts 1. What is the probability that the fourth term in Jacob's sequence is an integer?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{3}{4}$

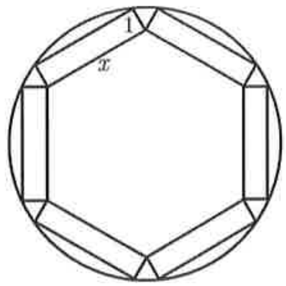
23. Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 **(B)** 40 **(C)** 60 **(D)** 160 **(E)** 320

24. Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

25. A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being end points of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x ?



(A) $2\sqrt{5} - \sqrt{3}$ **(B)** 3 **(C)** $\dfrac{3\sqrt{7} - \sqrt{3}}{2}$ **(D)** $2\sqrt{3}$

(E) $\dfrac{5 + 2\sqrt{3}}{2}$